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Strategic stiffening/cooling in the Ising game

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ABSTRACT

The dynamic noisy binary choice (Ising) game of forward-looking agents on a complete graph is analysed. It is shown that strategic considerations lead to effective interaction strengthening (noise reduction) as compared to the myopic game. We show that strategic agents are able to come to consensus in the wider range of noise values than myopic ones. Effective population dynamics with time-dependent probabilities reflecting this strategic stiffening/cooling effect is described.

1. Introduction

In the modern world an ability of society to resist various kinds of global risks is to a large extent determined by socio-economic factors. Social consensus and its robustness to an impact of information noise governs the effectiveness of authorities' regulating policies in different countries. Examples of such phenomena can be found in different fields: from financial stability (the phenomenon of bank runs) to global health issues (people's attitude towards vaccination). Despite clear distinctions, such processes have much in common. In particular, the key features of such systems are high connectivity and mutual influence of individuals. In addition, evolution of these systems is typically affected by various sources of endogenous and exogenous noise.

Social/economic interaction is believed to be one of the key factors underlying the complexity of the observed features of large social/economic systems and their evolution. Development of its quantitative description is thus known to be of high importance [1]. A particular strand of literature studying the effects of interactions in multiplayer games is devoted to the analysis of static equilibria and dynamical evolution in noisy discrete choice games, see e.g. [2–4]. The presence of noise makes it necessary to develop a probabilistic description of their equilibria as mixed strategy ones as well as of their dynamical evolution. From the game-theoretic point of view the corresponding static equilibria belong to the class of Quantal Response [5] or Boundedly Rational Nash [6] ones.

A formulation of a noisy discrete choice game includes specifying the strategy space (binary, ternary, etc.), noise distribution and topology of the underlying graph representing agents interaction. The predominant focus of the literature on noisy multiplayer discrete choice games is on the simplest binary choice case with a particular choice of noise distribution, the Gumbel one, and complete topology of an underlying graph [2]. Static equilibria and dynamical evolution for arbitrary noise distributions and complete graph topology was considered in [2,4], the case of Gumbel noise and several fixed topologies was analysed in [7]. The case of arbitrary noise distribution and random graph topology was discussed in [8–10].

The choice of the Gumbel distribution allows to draw deep parallels with statistical physics of magnetics. A useful description of its main relevant aspects can be found in e.g. [11,12]. A discussion of such parallels between noisy discrete choice games and statistical physics, most importantly on ordered/disordered phases and corresponding phase transitions and mean field dynamics, can be found in [13–15], see also an interesting discussion in [16]. In particular, both phase structure and dynamical evolution of the binary choice game on the complete graph with Gumbel noise is equivalent to those in the mean field Ising model [11,15]. Similar analogies played an important role in discussing formation of public opinion in sociodynamics/sociophysics [17,18].

It is, however, clear that the above-described equivalence between physical models and those of socio-economic systems cannot be universal. One particular aspect potentially distinguishing spins in magnetics from socio-economic agents is that the latter are, generally speaking, forward - looking, i.e. their decision can be based not only on the current system's state or its history, but can also be influenced by their forecast of possible system evolution in future. This aspect of human decision-

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making underlies an analysis of repeated games in game theory [19] and intertemporal effects in macroeconomics [20] where strategic behaviour is described through introducing an expectation of a sum of a stream of discounted future utilities/payoffs as a factor correcting the myopic decision making. Technically determining a locally optimal decision requires solving a dynamical programming problem through finding a solution of the corresponding Bellman equation, see e.g. [21]. The present paper builds upon the analysis of the myopic population dynamics of the Ising game on complete graph for arbitrary noise distributions presented in [4] by enhancing myopic decision making with forward-looking strategic considerations.

To the authors' knowledge this study is the first attempt to introduce strategic behaviour into the Ising or similar noisy discrete choice game. However, there is a series of papers in which strategic agents behaviour was considered in the context of evolutionary 2×2 bimatrix games. In particular, in [22] effects of strategic behaviour on equilibrium selection in a deterministic repeated 2×2 game was analysed. Several papers were devoted to multi-agent games in which at some random time points agents obtain possibility to change their strategies [23–25]. The main question addressed in these studies was effects of strategic behaviour on equilibrium selection. In myopic games of such kind agent decides to change its strategy only as a result of the analysis of the current population state. Forward-looking agents try to predict future evolution of the population and maximise their total utility on some horizon (finite or infinite). In particular, in [23] an existence of the evolutionary stable equilibrium in the noisy game of this kind. The analysis shows that this equilibrium converges to the myopic one when either the discount rate goes to infinity (representing decrease of the agents foresight) or the frequency of strategy revision declines. In [24] the variant of 2×2 coordination game with the same noisy component for all players was analysed. In this paper a dependence between agents' foresightedness and solution phase with respect to noise is outlined. Namely, when the noise amplitude is rather high, farsighted agents are act as myopic ones in the system with lower noise amplitude. In [25] the 2×2 coordination game where some fraction of agents was farsighted while the others – myopic. Foresight in this model means ability to make a two-step forecast. The convergence of the equilibrium to the Nash one was examined. It was shown that presence of sufficient fraction of forward-looking agents enables such kind of convergence. Results close to the previous one were also obtained in [26], however, heterogeneity of the agents foresight was achieved through variation of agents' discount rates.

The main goal of our paper is to describe a phenomenon of the effective interaction enhancement/noise reduction due to strategic considerations of agents in the Ising game. This strategic stiffening/cooling phenomenon has some qualitative similarities with the effect of strategic interactions described in [24].

It is known, that one of the fundamental properties of the myopic Ising game is existence of some kind of phase transition [2,4,7], i.e. there is a critical noise amplitude separating the solution space into two areas corresponding to disordered and ordered phases correspondingly. The disordered one, corresponding to high noise values, is characterised by zero mean equilibrium choice, while the ordered one is described by appearance of agent's (partial) consensus. The main result of the current study shows that when agents are forward-looking the dynamics converges to the consensus in larger domain than in the case of myopic game. I.e. solutions rearrangement appears at higher noise amplitudes. In other words, strategic considerations by forward-looking agents effectively reduce noise. This effect represents some kind of self-fulfilling expectations. When agent makes a decision he/she believes that consensus is preferred for the other players and this belief affects his current decision. This effect can have important consequences for the analysis of large-scale social phenomena such as dramatic opinion formation phenomena in sociodynamics [17].

The structure of the paper is the following.

In Section 2 we describe the Ising game considered in the paper. In particular, in paragraph 2.1 we present an overview of the existing results on myopic Ising game on complete graph. The paragraph 2.2 describes the dynamic discrete-time Ising game of forward-looking agents on complete network. In section 3 main results of the analysis are presented. Concluding remarks and outlook are presented in section 4. The paper includes several appendices. In the Appendix A we provide a detailed description of the solution of the Bellman equation for the value function. In the Appendix B we provide additional details on the stochastic simulation of the Ising game under consideration. In particular, in Appendix B.1 we provide additional detail on the procedure of stochastic simulations and in the Appendix B.2 - additional material on the effect of strategic stiffening/cooling on generic description of system dynamics in terms of ordered/disordered phases.

2. The Ising game

The dynamic Ising game studied in the present paper is formulated as follows. There are N agents placed in the vertices of a complete graph. A strategy space of each agent i, $i = 1, \ldots, N$, consists of two pure strategies $s_i = \pm 1$. The game lasts for T time periods $t = 1, \ldots T$ and at each time step t the full description of a system is given by a strategies configuration $\mathbf{s}(t) = (s_1(t), s_2(t), \ldots s_N(t))$. The evolution starts with some initial configuration $\mathbf{s}(0)$ and is assumed to be driven by strategy revisions $s_i(t-1) \to s_i(t)$ by one randomly chosen agent i per time step.

2.1. Myopic game

Let us first describe the standard myopic version of the Ising game [2, 4,7–10]. The process of myopic strategy revision by an agent i is based on assessing the value/utility $U_i^{\rm mp}(s_i, \mathbf{s}_{-\mathbf{i}}(t), \varepsilon_{s_i}(t))$ of choosing a strategy s_i

$$U_i^{\rm mp}(s_i, \mathbf{s}_{-i}(t), \varepsilon_{s_i}(t)) = J\left(\frac{1}{N}\sum_{j \neq i} s_j(t)\right) s_i + \varepsilon_{s_i}(t) \tag{1}$$

where $J > 0$ is a coupling constant determining the strength of conformity effect, $\mathbf{s}_{-i}(t) = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots s_N)$ is a condensed notation for a vector of strategies of neighbouring agents at time t and $\varepsilon_{s_i}(t)$ is a strategy - dependent random contribution. It is assumed that probability distributions for $\{\varepsilon_{s_i}\}$ are the same for all agents and are Gaussian $\mathcal{N}(0, \sigma \equiv 1/\beta)$. In what follows we will use a condensed notation

$$m_{-i} = \frac{1}{N}\sum_{j \neq i} s_j$$

so that

$$U_i^{\rm mp}(s_i, \mathbf{s}_{-i}(t), \varepsilon_{s_i}(t)) = U_i^{\rm mp}(s_i, m_{-i}(t), \varepsilon_{s_i}(t)) = J m_{-i}(t) + \varepsilon_{s_i}(t).$$

Let us introduce an explicit notation $\Phi_i(t)$ for the information set available for each agent at time t

$$\Phi_i(t) = \left(\mathbb{I}_i(t), s_i(t-1), m_{-i}(t), \varepsilon^i_{\pm 1}(t)\right), \tag{2}$$

where $\mathbb{I}_i(t)$ is an indicator function equal to 1 if at time point t player i gets a chance of strategy revision and to 0 otherwise. Using this notation we will denote myopic utility function as follows

$$U_i^{\rm mp}(s_i, \Phi_i(t)) \equiv \begin{cases} U_i^{\rm mp}(s_i, m_{-i}(t), \varepsilon_{s_i}(t)), & s_i = \pm 1, \ \text{if}\ \mathbb{I}_i(t) = 1 \\ U_i^{\rm mp}(s_i(t-1), m_{-i}(t), \varepsilon_{s_i(t-1)}(t)), & \text{if}\ \mathbb{I}_i(t) = 0 \end{cases}$$

The strategy of an agent i formed at time t is thus

$$s_i(t) = \begin{cases} \mathrm{argmax}_{s_i'} U_i^{\mathrm{mp}}(s_i', \Phi_i(t)) & \mathbb{I}_i(t) = 1 \\ s_i(t-1) & \mathbb{I}_i(t) = 0 \end{cases} \tag{3}$$

The process of evolution of $\mathbf{s}(t-1) \to \mathbf{s}(t)$ at time t is thus driven by strategy revision $s_i(t-1) \to s_i(t)$ of an agent i for whom $\mathbb{I}_i(t) = 1$.

After the process of strategy revision is completed the agents collect a vector of payoffs $\mathbf{w}(\mathbf{s}(t))$

$$\mathbf{w}(\mathbf{s}(t)) = \left(U_i^{\mathrm{mp}}(s_1(t), \Phi_1(t)), \ldots, U_N^{\mathrm{mp}}(s_N(t), \Phi_N(t))\right) \tag{4}$$

2.2. Strategic game

Myopic rationality expressed in (3) corresponds to the simplest possible type of behaviour indistinguishable from "rationality" of spin flips in Glauber dynamics. Analysis of multiperiod problems in economics and game theory is based on a notion of far-sighted (strategic) agents who take into account not only the locally existing circumstances as in (3), but also expectations related to possible subsequent evolution of both $\mathbf{s}(t)$ and the associated payoffs $\mathbf{w}(\mathbf{s}(t))$. For a far-sighted agent the choice utility is a sum of myopic U_i^{mp} and strategic U_i^{st} contributions

$$U_i(s_i, \Phi_i(t)) = U_i^{\mathrm{mp}}(s_i, \Phi_i(t)) + \gamma U_i^{\mathrm{st}}(s_i, \Phi_i(t)). \tag{5}$$

The expression for the strategic contribution is an expectation value of a discounted sum of future utilities

$$U_i^{\mathrm{st}}(s_i, \Phi_i(t)) = \mathbb{E}\left(\sum_{\tau=t+1}^{T} \gamma^{\tau-t-1} U_i^{\mathrm{mp}}(s_i(\tau), \Phi_i(\tau)) | s_i(t) = s_i, \Phi_i(t)\right) \tag{6}$$

where γ is a discounting factor and averaging is over a set of trajectories $\{\mathbf{s}(1) \to \mathbf{s}(2) \to \ldots \to \mathbf{s}(T)\}$ generated by two sources of randomness: the first is in the randomness in assigning the possibility of strategy revision to an agent or one of the agents in his neighbourhood, the second is in random strategy-dependent contributions to utility $\{\varepsilon_{si}^{i}\}$. It should be stressed, that in fact there is one more argument in $U_i^{\mathrm{st}}(\cdot)$ – the policy function determining how agents will choose $\mathbf{s}(\tau)$, $\tau > t$. But we omit this argument to simplify the notations.

The strategy revision process generalising the myopic one described in (3) is now described

$$s_i(t) = \begin{cases} \mathrm{argmax}_{s_i'} U_i(s_i', \Phi_i(t)) & \mathbb{I}_i(t) = 1 \\ s_i(t-1) & \mathbb{I}_i(t) = 0 \end{cases} \tag{7}$$

where the utility $U_i(s_i', \Phi_i(t))$ contains both myopic and strategic contributions, see (5,6). The payoff for the resulting $\mathbf{s}(t)$ is, as before, given by Eq. (4).

The process of decision making taking into account future system trajectories is conveniently related to a value function $V_i(s_i(t), \Phi_i(t), t)$

$$V_i(s_i(t), \Phi_i(t), t) = U_i^{\mathrm{mp}}(s_i(\underbrace{t}_{\text{instantaneous payoff}}), \Phi_i(t)) + . \\ \underbrace{\gamma \mathbb{E}\left(\max_{\substack{s_i(t+1), \ldots, \\ s_i(T) \in U(t)}} \sum_{\tau=t+1}^{T} \gamma^{\tau-t-1} U_i^{\mathrm{mp}}(s_i(\tau), \Phi_i(\tau)) s_i(t), \Phi_i(t)\right)}_{\text{discounted future payoffs}} \tag{8}$$

A key difference between the value function (8) and the utility (5) is that in the second term in (8) one averages over trajectories with utility-maximising choices for $s_i(\tau)$ for times at which the agent i gets a chance of strategy revision. In other words, in (8) the particular policy function is used.

The optimal strategy $s_i^*(t)$ and the optimal value function $V_i^*(\Phi_i(t), t)$ for a far-sighted agent are therefore described as

$$s_i^*(t) = \underset{s_i'}{\mathrm{argmax}}\left[V_i(s_i', \Phi_i(t), t)\right], \quad V_i^*(\Phi_i(t), t) = \max_{s_i'}\left[V_i(s_i', \Phi_i(t), t)\right] \tag{9}$$

It is important to stress that due to the presence of strategic contribution in (8) the optimal strategy found through (9) can be different from the myopic greedy one defined by (3).

The optimal value function $V_i^*(\Phi_i(t), t)$ satisfies the Bellman optimality equation

$$V_i^*(\Phi_i(t), t) = \max_{s_i}\left[U_i^{\mathrm{mp}}(s_i(t), \Phi_i(t)) + \right. \\ \left. \gamma \mathbb{E}\left(V_i^*(\Phi_i(t+1), t+1) | s_i(t), \Phi_i(t)\right)\right] \tag{10}$$

In what follows we still use a special notation for the strategic contribution to the value function in (8)

$$\begin{aligned} Q_i(s_i(t), \Phi_i(t), t) &= \\ &= \mathbb{E}\left(\max_{\substack{s_i(t+1), \ldots, \\ s_i(T) \in U(t)}} \sum_{\tau=t+1}^{T} \gamma^{\tau-t-1} U_i^{\mathrm{mp}}(s_i(\tau), \Phi_i(\tau)) | s_i(t), \Phi_i(t)\right) \\ &= \mathbb{E}\left(V_i^*(\Phi_i(t+1), t+1) | s_i(t), \Phi_i(t)\right) \end{aligned} \tag{11}$$

Using the definition (11) the expression (8) can be written as

$$V_i(s_i, \Phi_i(t), t) = Jm_{-i}(t) s_i + \gamma Q_i(s_i, \Phi_i(t), t) + \varepsilon_{s_i}^{i} \tag{12}$$

As seen from eq. (11), calculation $Q_i(s_i(t), \Phi_i(t), t)$ can proceed by solving the Bellman eq. (10) for the expected value of the optimal value function $\mathbb{E}\left(V_i^*(\Phi_i(t+1), t+1) | s_i(t), \Phi_i(t)\right)$. It turns out that for the considered case of the complete graph it is possible to construct its exact numerical solution. The corresponding details are provided in the Appendix A.

3. Results

With the elements of the game at place let us first analyse whether strategic thinking pays off, i.e. whether it increases utility collected by the players in the course of the game. Let us remind that at a given time moment an agent can be

- passive, collecting the instantaneous utility (payoff)

$$U_i^{\mathrm{mp}}(t) \equiv U_i^{\mathrm{mp}}\left(s_i(t-1), m_{-i}(t), \varepsilon_{s_i(t-1)}\right)$$

- active, collecting the instantaneous utility (payoff)

$$U_i^{\mathrm{mp}}(t) \equiv U_i^{\mathrm{mp}}\left(s_i(t), m_{-i}(t), \varepsilon_{s_i(t)}\right)$$

The difference between the two cases is that an active agent collects utility corresponding to the new choice that, in turn, includes strategic considerations. The simplest characteristics of the process of utility generation is an averaged over an ensemble of system's simulated trajectories mean agents' payoff

$$\langle U(t) \rangle = \frac{1}{N} \sum_i \langle U_i^{\mathrm{mp}}(t) \rangle, \tag{13}$$

The resulting dependence is shown, for several values of γ and σ, in Fig. 1. The details of the numeric simulations are provided in Appendix B.

From the results shown in Fig. 1 we see that including considerations related to future payoffs into the decision contour leads to increase of the average collected utility as compared to the myopic game when

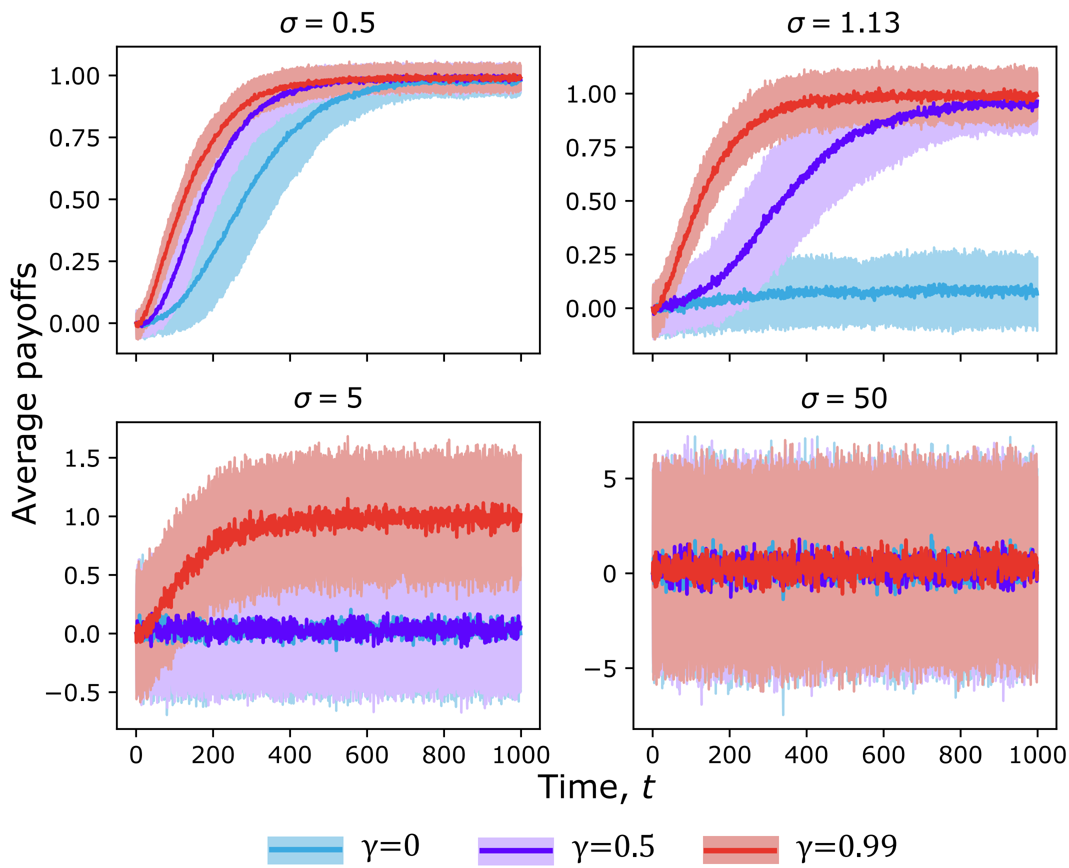

Fig. 1. Evolution of the averaged over an ensemble of system's simulated trajectories mean agents' payoff $U(t)$. The solid lines show the position of $U(t)$, when averaging is performed over 100 simulations. Areas represent the size of the corresponding mean agents' payoff standard deviations.

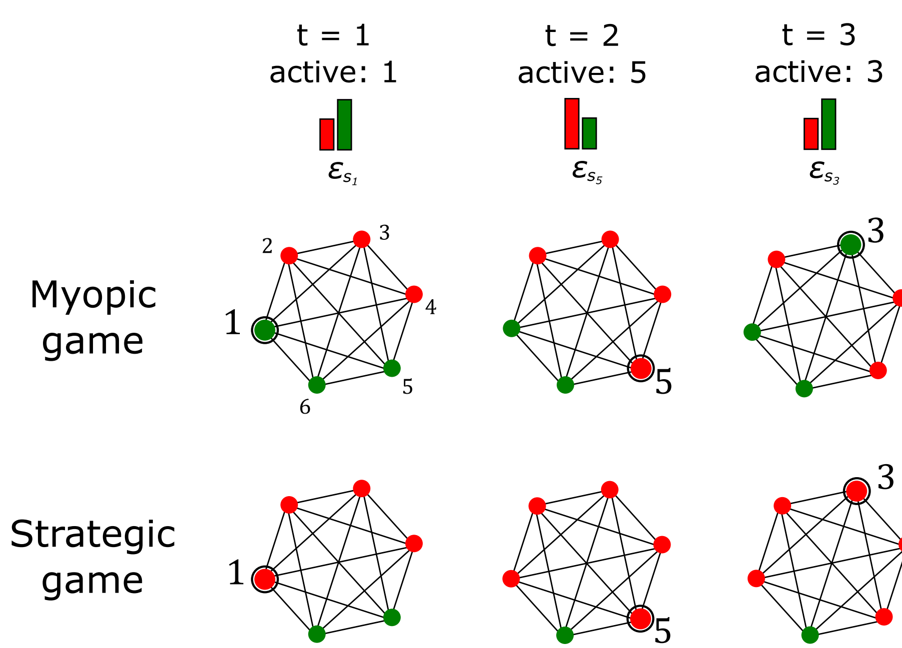

Fig. 2. Illustrative example of the myopic and strategic games of 6 players. Green colour denotes the $+1$ choice, red colour – the -1 one. The rectangles in the top show relative values of the noise ε_{-1} and ε_{+1}. The circled node represents a position of the current active player.

noise is not too high. The magnitude of the effect increases with γ. The upper-right panel of the figure corresponds to the point $\sigma = 1.13$ at which the myopic phase transition takes place. From the presented plot we see that for the myopic game ($\gamma = 0$) at this point $U(t)$ starts to fluctuate around zero. A notable property of the strategic game is in the shift of transition point to the higher noise values. Moreover, the higher is the value of γ, the later does this transition occur.

In what follows we present a detailed description of this phenomenon. However, to clarify the idea, let us preface this discussion by considering an illustrative example. Let us consider the case of $N = 6$ players placed in the vertices of the $N = 6$ - nodes clique. In Fig. 2 we compare two games: the myopic and strategic ones. Suppose, that we analyse three time periods, for both games choose the same agents to be active at every time point and that realisations of ε_{s_i} are also the same. In other words, we analyse two games with equivalent sources of randomness. We also suppose that initially (at $t = 0$) agents 1, 5, 6 play $+1$ and agents 2, 3, 4, oppositely, play -1.

- At $t = 1$ the agent 1 is active. For him ε_+ is significantly higher than ε_-. Therefore, in the myopic game he/she will choose $s_1(1) = +1$. In the strategic game agent 1 accounts not only for the current system's state and noise values, but also analyses what will happen next. He/she understands, that if he/she will choose -1 the system will be much closer to the consensus state -1 (four players against two). Therefore, he/she understands that while currently losing a part of the “noisy” utility he/she may receive much more from the future consensus. So, the probability that he/she will choose -1 is high. Assume, that he/she chooses $s_1(1) = -1$.
- At $t = 2$ the agent 5 is active. For him/her ε_- is significantly higher than ε_+. We have the same incentives for the agent 5: it is better to choose -1. However, in the strategic case mean agents payoff is higher, as they now closer to the consensus, i.e. agent 1 was right to choose -1 at $t = 1$.
- At $t = 3$ the agent 3 is active. For him/her ε_+ is significantly higher than ε_-. The description of agent 3's choice is absolutely the same as the one of agent 1 at $t = 1$, however, the agent 3 appears to be in better conditions, as the system is closer to the consensus state.

From this description it is clear, that strategic players have more freedom in their decisions and, therefore, it is easier for them to find the almost consensus state.

The preceding description of the evolution of a strategy set $\mathbf{s}(t)$ was based on an explicit usage of (1) through explicit generation of ensembles $\{\varepsilon_{\pm 1}\}$ for each of the time steps $t \in 1 \ldots T$ for each episode and subsequent averaging over episodes. The result of this averaging can be expressed through considering probabilities $\{p_{s_i}(m_{-i}, t)\}$ of choosing, for an agent i, a strategy s_i for given m_{-i} at time t:

$$\begin{aligned} p_{s_i}(m_{-i}, t) &= \text{Prob}[V_i(s_i, \Phi_i(t), t) > V_i(-s_i, \Phi_i(t), t)] \\ &= \text{Prob}\left[\varepsilon_{-s_i} - \varepsilon_{s_i} < 2Jm_{-i}s_i + \gamma\Delta Q_i(s_i, m_{-i}, t)\right] \\ &= F_<(2Jm_{-i}s_i + \gamma\Delta Q_i(s_i, m_{-i}, t)) \end{aligned} \tag{14}$$

where

$$\Delta Q_i(s_i, m_{-i}, t) = Q_i(s_i, m_{-i}, t) - Q_i(-s_i, m_{-i}, t) \tag{15}$$

and

$$F_<(x) = \text{Prob}\left[\varepsilon_{-s_i} - \varepsilon_{s_i} < x\right] \tag{16}$$

Here by denoting $Q_i(s_i, m_{-i}, t)$ we assume $Q_i(s_i, \Phi_i(t), t)$ for the case when $\mathbb{I}_i(t) = 1$. In the myopic limit of $\gamma = 0$ we get

$$\begin{aligned} p^{\text{mp}}_{s_i}(m_{-i}) &= \text{Prob}\left[U^{\text{mp}}_i(s_i, m_{-i}) > U^{\text{mp}}_i(-s_i, m_{-i})\right] \\ &= \text{Prob}\left[\varepsilon_{-s_i} - \varepsilon_{s_i} < 2Jm_{-i}s_i\right] \end{aligned} \tag{17}$$

For the probit noise $\mathcal{N}(0, \sigma \equiv 1/\beta)$ considered in the paper

$$p_{s_i}(m_{-i}) = \frac{1}{2}\left(1 + \text{ erf}\left[\beta Jm_{-i}s_i + \frac{1}{2}\beta\gamma\Delta Q_i(s_i, m_{-i}, t)\right]\right) \tag{18}$$

The key difference between the choice probabilities (17) and (14) is that the myopic probabilities (17) have no explicit time dependence while those in (14) are, on the contrary, explicitly time-dependent due to the time dependence of the strategic contribution $\Delta Q_i(s_i, m_{-i}, t)$ to the choice utility.

For the considered case of the Ising game on a complete graph the probabilities in (14) are the same for all nodes. In addition, in the large N limit

$$m_{-i}(t) = \frac{1}{N}\sum_{j=1}^{N} s_j(t) - \frac{s_j}{N} \simeq m(t) = \frac{1}{N}\sum_{j=1}^{N} s_j(t), \tag{19}$$

This means that the index i in (14,17) can be omitted so that the process of decision making can be described by one universal probability distribution $p_s(m, t)$:

$$\begin{aligned} p_s(m, t) &= F_<(2Jms + \gamma\Delta Q(s|m, t)) \\ &= \frac{1}{2}\left(1 + \text{ erf}\left[\beta Jms + \frac{1}{2}\beta\gamma\Delta Q(s, m, t)\right]\right) \end{aligned} \tag{20}$$

In what follows we will concentrate on the analysis of

$$p_+(m, t) \equiv p_{s=1}(m, t) = \frac{1}{2}\left(1 + \text{erf}\left[\beta Jm + \frac{1}{2}\beta\gamma\Delta Q(1, m, t)\right]\right) \tag{21}$$

A detailed description of the behaviour of the function $\Delta Q(1, m, t)$ is presented in the Appendix C. In particular, a transparent interpretation of the effects of strategic contribution is possible for discount parameters $\gamma < 0.85$ (for $\sigma \geq 1$). In this case the function $\Delta Q(1, m, t)$ is to a very high accuracy linear in m:

$$\Delta Q(1, m, t) = c(t)m, \tag{22}$$

with $c(t) > 0$ staying practically constant almost up to $t = T$ and rapidly falling to $c(T) = 0$ at the very end. In fact, $c(t) \equiv c\left(t, \gamma, \sigma \equiv \frac{1}{\beta}\right)$, but we omit γ and σ to simplify the notations. Then from eqs. (21,22) we get for $p_+(m, t)$

$$p_+(m, t) \approx F_<[(2J + \gamma c(t))m)] \tag{23}$$

For the considered case of probit noise the expression (23) takes the form

$$p_+(m, t) \equiv p_{s=1}(m, t) = \frac{1}{2}\left(1 + \text{erf}\left[\beta\left(J + \gamma\frac{c(t)}{2}\right)m\right]\right) \tag{24}$$

In the myopic limit the game dynamics is controlled by the product βJ quantifying the balance of the strength of rigidity represented by the interaction scale J and noise represented by $\beta = 1/\sigma$. From eq. (24) we see that taking into account strategic contribution to utility leads to the following modification:

$$(\beta J)_{\text{eff}} = \beta J\left(1 + \gamma\frac{c(t)}{2J}\right) \tag{25}$$

As at all $t < T$ the coefficient $c(t)$ is positive, the new balance between interaction and noise is shifted to $(\beta J)_{\text{eff}} > \beta J$. This effect of far-sightedness can therefore alternatively be interpreted as

- **Strategic stiffening**, i.e. an increase of interaction strength J

$$J \quad \rightarrow \quad J\left(1 + \frac{\gamma c(t)}{2J}\right); \tag{26}$$

- **Strategic cooling**, i.e. a reduction of noise

$$\beta \quad \rightarrow \quad \beta\left(1 + \frac{\gamma c(t)}{2J}\right) \tag{27}$$

or some combinations thereof corresponding to some particular multiplicative partitions of the second factor in the right hand side of (26, 27).

The above-presented arguments are supported by results of computation of $p_+(m,t)$ based on the exact numerical solution of the Bellman equation described in the Appendix A shown, for different times, in Fig. 3. We see, that at the final time point $t = T$ the function $p_+(m,T)$ is purely myopic so that is behaviour is determined by the myopic scale βJ. On the contrary, starting from the initial moment $t = 1$ the function $p_+(m,t)$ is markedly different from its myopic limit showing significant noise reduction.

A direct illustration of the phenomenon of strategic stiffening/cooling is probided by studying the temporal evolution of the scale $(\beta J)_{\text{eff}}(t)$ which is generically defined by

$$p_+(m,t) = \frac{1}{2}\left(1 + \text{erf}\left[(\beta J)_{\text{eff}}(t) m\right]\right) \tag{28}$$

The time dependence of $(\beta J)_{\text{eff}}(t)$ is shown in Fig. 4. in which one directly observes noise reduction due to strategic contribution to decision-making and, in particular, its dependence on the discount parameter γ, cf. Eq. (25).

In the considered case of the complete graph topology the dynamics of the Ising game is completely characterised by the evolution of $m(t)$. Let us describe this evolution in terms of a population game with time-dependent rates computed from the time-dependent probabilities $p_+(m,t)$ taking into account strategic effects in decision making. The state of this population game is, at each t, fully described by the number of agents $N_\pm(t)$ equipped with the strategies $s = \pm 1$ respectively so that

$$m(t) = \frac{N_+(t) - N_-(t)}{N} \tag{29}$$

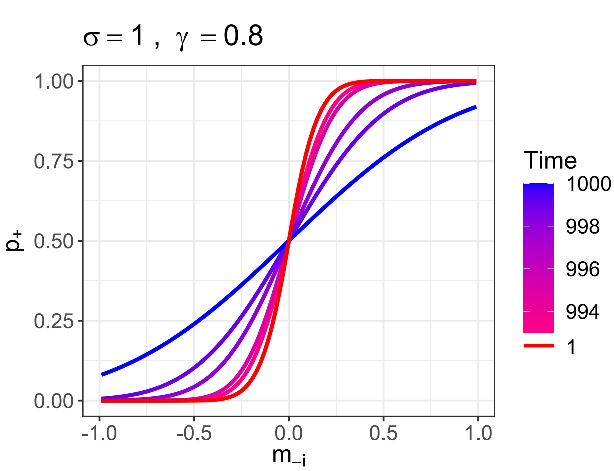

Fig. 3. Temporal evolution of the probability $p_+(m,t)$.

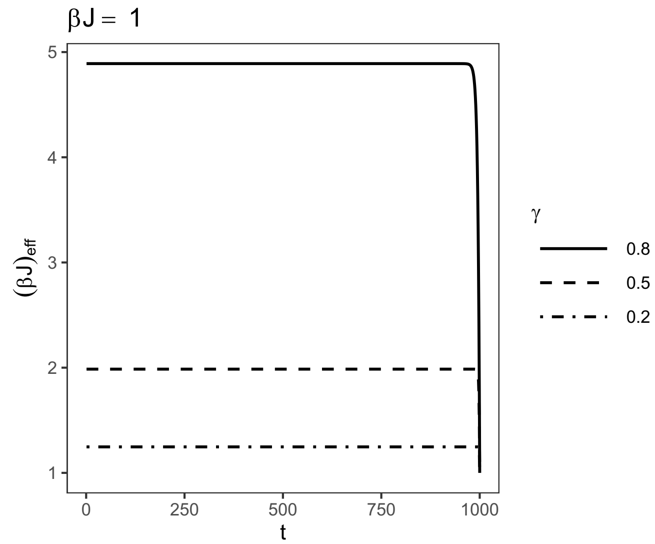

Fig. 4. Temporal evolution of $(\beta J)_{\text{eff}}(t)$ in different time points t and for different parameter γ.

The evolution is then fully defined by the state- and time-dependent transition rates $\lambda_\pm(N_\pm|t)$ describing probabilities of a strategy flip $s \to -s$ of a randomly chosen agent

$$\lambda_\pm(N_\pm|t) = \text{Prob}[M(t+1) = M(t) \pm 2] \tag{30}$$

where $M = N_+ - N_-$. We have

$$\lambda_\pm(N_\pm|t) = \frac{N_\mp(t)}{N} p_\pm(m(t),t) = \frac{1 \mp m(t)}{2} p_\pm(m,t) \tag{31}$$

The evolution of an average aggregate choice is then described by the following equation

$$\begin{aligned}\langle m(t+1) - m(t)|m(t)\rangle &= \frac{2}{N}\lambda_+ + \left(-\frac{2}{N}\right)\lambda_- = \\ &= \frac{1}{N}\left[(2p_+(m(t),t) - 1) - m(t)\right]\end{aligned} \tag{32}$$

where we have used the fact that $p_+(m,t) + p_-(m,t) = 1$ and the notation $\langle \cdot | m(t) \rangle$ means expectation value conditional on $m(t)$. To describe the evolution of $\langle m(t) \rangle$ one needs the expression not for the conditional expectation of the $\Delta m(t) = m(t+1) - m(t)$ value, but for the unconditional one $\langle \Delta m(t) \rangle$. This estimate can be obtain by application of the expectation operation to the both right- and left-hand sides of Eq. (32), i.e.

$$\langle m(t+1) - m(t) \rangle = \frac{1}{N}\langle [(2p_+(m(t),t) - 1) - m(t)] \rangle.$$

As very rough approximation we can put expectation operator inside the right-hand side function operator:

$$\langle m(t+1) - m(t) \rangle \approx \frac{1}{N}\left[(2p_+(\langle m(t) \rangle, t) - 1) - \langle m(t) \rangle\right]. \tag{33}$$

This approximation is assumed to be close to the exact solution for the low noise phase and close to equilibrium states. Using Eq.(33) we can write the expression for the evolution of the $\langle m(t) \rangle$ as follows

$$\langle m(t+1) \rangle \approx \langle m(t) \rangle + \frac{1}{N}\left[(2p_+(\langle m(t) \rangle, t) - 1) - \langle m(t) \rangle\right] \tag{34}$$

In Fig. 5 we compare evolution of simulated $m(t)$ with that described by Eq. (34). We see that a very good agreement between the two is observed.

Characteristic regimes of the evolution of the configuration $\mathbf{s}(t)$ and, in particular, of its aggregated characteristic $m(t)$ described by Eq. (32) are significantly determined by the relative strength of conformity and noise effects parametrised by the constants J and $\beta = 1/\sigma$ respectively. These regimes can be classified by recalling that a static myopic Ising game is characterised by a phase transition between ordered and disordered phases with $m \neq 0$ at $(\beta J) > (\beta J)^{\text{mp}}_{\text{crit}}$ and $m = 0$ at $(\beta J) < (\beta J)^{\text{mp}}_{\text{crit}}$ correspondingly [2,4] where for the considered case of probit noise one has [10].

$$(\beta J)^{\text{mp}}_{\text{crit}} = \frac{\sqrt{\pi}}{2} \tag{35}$$

or, equivalently,

$$\sigma^{\text{mp}}_{\text{crit}} = \frac{2J}{\sqrt{\pi}}. \tag{36}$$

Assuming for definiteness that strategic effects manifest themselves in the form of strategic cooling we see from (27) that the perceived effective level of noise $\sigma_{\text{eff}}(t)$ is related to the bare one σ by

$$\sigma_{\text{eff}}(t) = \sigma\left(\frac{2J}{2J + \gamma c(t)}\right) \tag{37}$$

so, as long as $\sigma_{\text{eff}}(t) < \sigma^{\text{mp}}_{\text{crit}}$, i.e.

$$\sigma \;<\; \sigma^{\text{mp}}_{\text{crit}}\left(1 + \frac{\gamma c(t)}{2J}\right) \tag{38}$$

the system will find itself in the low-noise phase. Of special interest here is of course the time-dependent interval

$$\sigma^{\text{mp}}_{\text{crit}} \;<\; \sigma \;<\; \sigma^{\text{mp}}_{\text{crit}}\left(1 + \frac{\gamma c(t)}{2J}\right) \tag{39}$$

in which the effective reduction of noise pushes the system into the low-noise phase for the values of bare noise strength σ at which the myopic phase is the high-noise one. A detailed illustration of this phenomenon is given in Appendix B.2.

4. Conclusion and outlook

Let us formulate once again the main results obtained in the present paper.

- The noisy dynamic binary choice game (Ising game) of forward-looking agents on complete graph was analysed.
- It was shown that strategic considerations lead to enhancement of average local utility as compared to the myopic game.
- A phenomenon of strategic stiffening/cooling of interaction enhancement/noise reduction due to farsighted strategy formation was described.

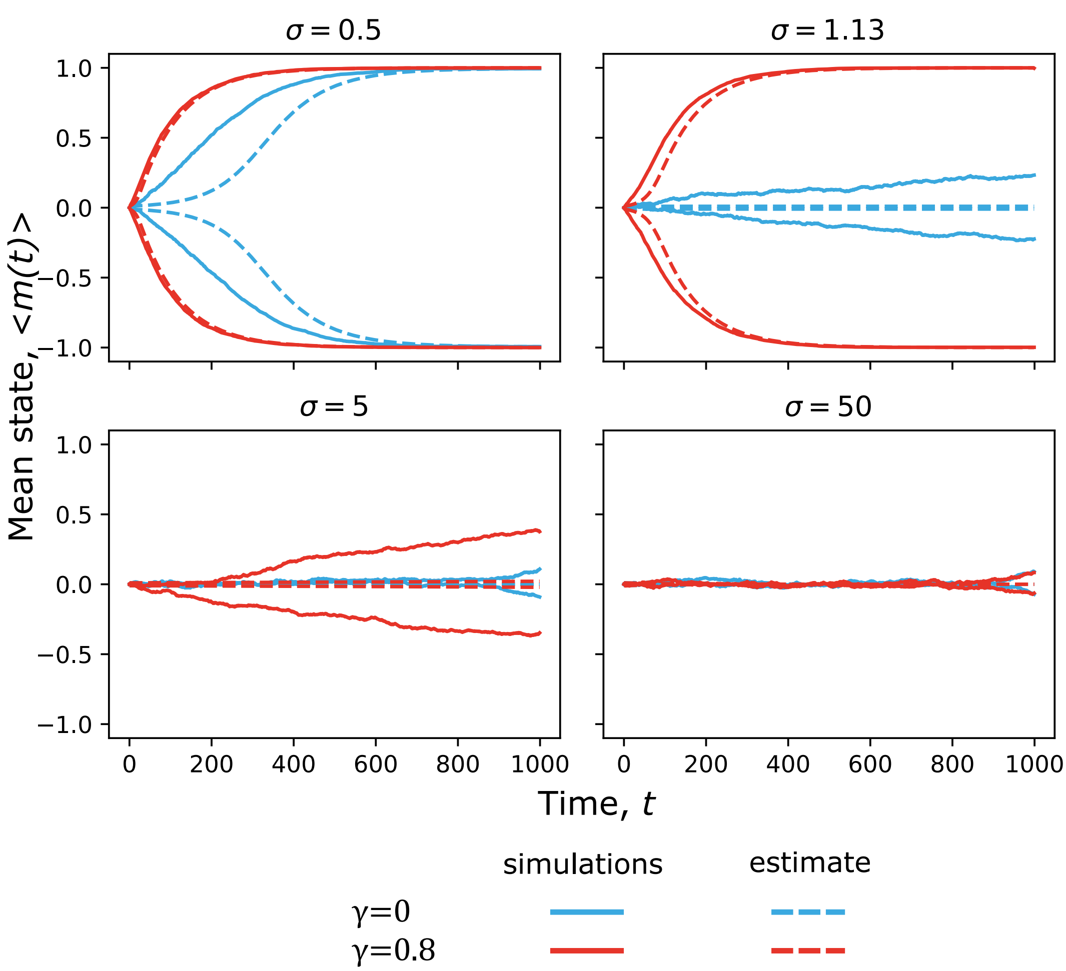

Fig. 5. Evolution of the $\langle m(t)\rangle$ in the numerical simulations and obtained using Eq. (34) for the myopic case ($\gamma = 0$) and for the case of $\gamma = 0.8$.

- A modified population dynamics with time-dependent choice probabilities was formulated.
- The effect of strategic stiffening/cooling can have important consequences for describing the collective dynamics effects in the Ising type models of opinion formation.

The two important problem for the next phase of analysis we see are:

- Studying effects of agent's farsightedness in the Ising game on graphs with nontrivial topologies. In this case exact solution of the corresponding Bellman equation will no longer be possible so one will have to turn to finding its approximate solution.
- Studying reinforcement learning in the Ising game in the spirit of [27].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Solution of Bellman equation

To account for strategic component in the utility function one needs to calculate values $Q_i(\pm 1, \Phi_i(t), t)$ for every possible value of $m_{-i}(t)$ and $t \in [1, \ldots, T]$, or the table $\mathbf{Q}$ (see Table B.2 in the appendix Appendix B). As it follows from (11) to do these calculations one needs to obtain values of the optimal value function $V_i^*(\Phi_i(t+1), t+1 | s_i(t), \Phi_i(t))$ for every possible tuple $\Phi_i(t+1)$ and t values. Common method of computation of the optimal value function for the problems with finite time horizon is backward induction. In this approach computation of the value function values begins from the final time point.

In the considered game tuple $\Phi_i(t+1)$ contains realisations of random contributions ε^i_{+1} and ε^i_{-1}. Therefore, in general one should treat optimal value function $V_i^*(\cdot)$ as having continuous arguments. However, to obtain $Q_i(\pm 1, \Phi_i(t), t)$ one needs only expected value of the $V_i^*(\Phi_i(t+1), t+1 | s_i(t), \Phi_i(t))$. Thus, there are two important notes about random contributions assessment. First, agent's decision is driven not by the values ε^i_{+1} and ε^i_{-1} itself, but by the following comparison:

$$Jm_{-i}(t) + \gamma Q_i(+1, \Phi_i(t), t) + \varepsilon^i_{+1} \text{vs.} -Jm_{-i}(t) + \gamma Q_i(-1, \Phi_i(t), t) + \varepsilon^i_{-1}$$

Second, as random contributions are additive and have zero mean they cancel out when values of $V_i^*(\cdot)$ are calculated. Therefore, to assess $Q_i(\pm 1, \Phi_i(t), t)$ instead of the pair $\varepsilon^i_{+1},\ \varepsilon^i_{-1}$ one needs to know only realization of the random value $\mathbb{I}_i^+$ which is defined as follows:

$$\mathbb{I}_i^+(t) = \mathbb{I}\left[Jm_{-i}(t) + \gamma Q_i(+1, \Phi_i(t), t) + \varepsilon^i_{+1} \geq -Jm_{-i}(t) + \gamma Q_i(-1, \Phi_i(t), t) + \ \varepsilon^i_{-1}\right] \tag{A.1}$$

Let $\tilde{\Phi}(t)$ be the following tuple

$$\tilde{\Phi}(t) = \left(\mathbb{I}_i(t), s_i(t-1), m_{-i}(t), \mathbb{I}_i^+(t)\right).$$

Then the Bellman table $\tilde{\mathbf{V}}$ with elements $\tilde{V}_i\left(\tilde{\Phi}(t), t\right)$ has structure defined in Table A.1. To enable the backward induction, it should be noted that $Q_i(\pm 1, \Phi_i(T), T) = 0$.

The exact way of calculation of the $Q_i(\pm 1, \Phi_i(t), t)$ values used in the definition of $\tilde{V}_i\left(\tilde{\Phi}(t), t\right)$ is described below. The backward induction is also illustrated in Fig. A.6.

Table A.1

Structure of the table $\tilde{\mathbf{V}}$.

$\tilde{\mathbf{V}}=$								
$\tilde{\Phi}(t)$				Time t				
$\mathbb{I}_i(t)$	$m_{-i}(t)$	$s_i(t-1)$	$\mathbb{I}_i^+(t)$	1	2	...	$T-1$	T
1	$-(N-1)/N$	1	1					
...	...	...	...					
1	$(N-1)/N$	1	1	$Jm_{-i}(t) + \gamma Q_i(+1, \Phi_i(t), t)$				
1	$-(N-1)/N$	-1	1					
...	...	...	...					
1	$(N-1)/N$	-1	1					
	$-(N-1)/N$	1	0					
...	...	...	...					
1	$(N-1)/N$	1	0	$-Jm_{-i}(t) + \gamma Q_i(-1, \Phi_i(t), t)$				
1	$-(N-1)/N$	-1	0					
...	...	...	...					
1	$(N-1)/N$	-1	0					
0	$-(N-1)/N$	1	1					
...	...	...	...					
0	$N-1$	1	1	$Jm_{-i}(t)s_i(t-1)+$ $\gamma Q_i(s_i(t-1), ,\Phi_i(t), ,t)$				
0	$-(N-1)$	-1	1					
...	...	...	...					
0	$N-1$	-1	1					

Table A.1 (*continued*)

$\tilde{\mathbf{V}}=$								
$\tilde{\Phi}\,(t)$				Time t				
$\mathbb{I}_i(t)$	$m_{-i}(t)$	$s_i(t-1)$	$\mathbb{I}_i^+(t)$	1	2	...	$T-1$	T
0	$-(N-1)$	1	0					
...	...	...	...					
0	$N-1$	1	0					
0	$-(N-1)$	-1	0					
...	...	...	...					
0	$N-1$	-1	0					

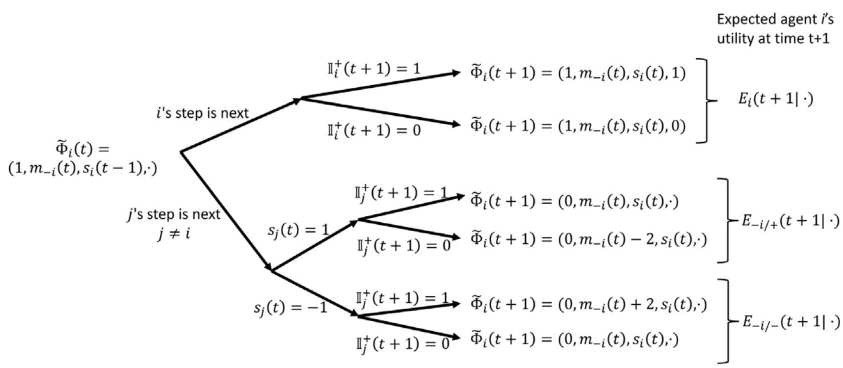

Fig. A.6. Illustration of the backward induction procedure used for the expected value $\mathbb{E}\left(\underset{i}{\tilde{V}}\left(\underset{i}{\tilde{\Phi}}(t+1),t+1\right)|s_i(t),\underset{i}{\tilde{\Phi}}(t)\right)$ calculation.

$$Q_i(s_i(t),\Phi_i(t),t)=\mathbb{E}\left(V_i^*(\Phi_i(t+1),t+1),s_i(t),\Phi_i(t)\right)=\mathbb{E}\left(\tilde{V}_i(\tilde{\Phi}_i(t+1),t+1),s_i(t),\tilde{\Phi}_i(t)\right)=P(\mathbb{I}_i(t+1)=1)\underbrace{E_i(t+1,s_i(t),m_{-i}(t))}_{\text{if } i'\text{s step is next}}+\underbrace{E_{-i/+}(t+1,s_i(t),m_{-i}(t))}_{\substack{\text{if some other player's step}\\ \text{is next and this player}\\ \text{has }+1\text{ strategy at time } t}}+$$

$$P(\mathbb{I}_i(t+1)=0)P(\mathbb{I}_j(t+1)=1,j\neq i,s_j(t)=-1)\times\underbrace{E_{-i/-}(t+1,s_i(t),m_{-i}(t))}_{\substack{\text{if some other player's step}\\ \text{is next and this player}\\ \text{has}-1\text{ strategy at time } t}},$$

where

$$P(\mathbb{I}_i(t+1)=1)=\frac{1}{N},\quad P(\mathbb{I}_i(t+1)=0)=1-\frac{1}{N},\quad P(\mathbb{I}_j(t+1)=1|j\neq i,s_j(t)=+1)=\frac{1}{2}\left(1+\frac{m_{-i}(t)}{N-1}\right)$$
$$P(\mathbb{I}_j(t+1)=1|j\neq i,s_j(t)=-1)=\frac{1}{2}\left(1-\frac{m_{-i}(t)}{N-1}\right)$$

$E_i(\cdot)$, $E_{-i/+}(\cdot)$ ($E_{-i/-}(\cdot)$) are player i's expected payoffs respectively in the cases when his step or his neighbour playing $+1$ (-1) step is next. Calculation of these values is described below.

$$\begin{aligned}E_i(t+1|s_i(t),m_{-i}(t)) &= \mathbb{E}\left[\left(\tilde{V}_i((1,m_{-i}(t),s_i(t),1),t+1)+\varepsilon^i_+\right)\mathbb{I}^+_i(t+1)+\left(\tilde{V}_i((1,m_{-i}(t),s_i(t),0),t+1)+\varepsilon^i_-\right)(1-\mathbb{I}^+_i(t+1))\right]\\ &= p^i_+(t+1)\tilde{V}_i((1,m_{-i}(t),s_i(t),1),t+1)+\left(1-p^i_+(t+1)\right)\tilde{V}_i((1,m_{-i}(t),s_i(t),0),t+1)+\mathbb{E}\left[\left(\varepsilon^i_+-\varepsilon^i_-\right)\mathbb{I}^+_i(t+1)\right],\end{aligned} \tag{A.2}$$

$$E_{-i/+}(t+1|s_i(t),m_{-i}(t))=p^j_+(t+1)\tilde{V}_i((0,m_{-i}(t),s_i(t),1),t+1)+\left(1-p^j_+(t+1)\right)\tilde{V}_i((0,m_{-i}(t)-2,s_i(t),1),t+1), \tag{A.3}$$

$$E_{-i/-}(t+1|s_i(t),m_{-i}(t))=p^j_+(t+1)\tilde{V}_i((0,m_{-i}(t)+2,s_i(t),1),t+1)+\left(1-p^j_+(t+1)\right)\tilde{V}_i((0,m_{-i}(t),s_i(t),1),t+1), \tag{A.4}$$

where $p^i_+(t+1)$ is the probability of the event that player i chooses $+1$ (the same object as defined in (14)):

$$p^i_+(t+1)=\mathrm{Prob}\left[\varepsilon^i_- -\varepsilon^i_+<\tilde{V}_i((1,m_{-i}(t),s_i(t),1),t+1)-\tilde{V}_i((1,m_{-i}(t),s_i(t),0)\right]=\mathrm{Prob}\left[\varepsilon^i_- -\varepsilon^i_+<2Jm_{-i}+\Delta Q_i(+1,\Phi_i(t),t+1)\right].$$

$p^j_+(t+1)$ in (A.3) and (A.4) has the similar definition. However, note that $m_{-j}(t)=m_{-i}(t)+s_i(t)-s_j(t)$.

Appendix B. Numerical simulations

B.1. Methodology and algorithm

The procedure of the numerical simulations can be divided into two stages: the preliminary and game one. On the first (preliminary) stage the parameters N, T, γ, J and σ are initialized and the table $\mathbf{Q}$ containing values $Q_i(s_i(t),\Phi_i(t),t)$ (see expression (11)) is calculated (see Table B.2). This table can be obtained directly from the Bellman table which is calculated in the Appendix Appendix A. As we study the model on the complete graph all the agents are absolutely the same, therefore, all agents actions are driven by the same $\mathbf{Q}$ table.

Table B.2
Structure of the table **Q**.

	$m_{-i}(t)$	$s_i(t)$	Time t				
			1	2	...	$T-1$	T
	$-(N-1)/N$	1					
	$-(N-3)/N$	1					
	...	...					
	$(N-3)/N$	1					
	$(N-1)/N$	1					
Q=	$-(N-1)/N$	-1	$Q_i(s_i(t),\Phi_i(t),t)$				
	$-(N-3)/N$	-1					
	...	...					
	$(N-3)/N$	-1					
	$(N-1)/N$	-1					

The game stage proceeds as follows. At each time step t one randomly chosen agent i gets a chance of a strategy revision $s_i(t-1)\to s_i(t)$ through a mechanism described in the section 2. As a result, the previous configuration $\mathbf{s}(t-1)$ is updated so that $\mathbf{s}(t)=(\ldots,s_{i-1}(t-1),s_i(t),s_{i+1}(t-1)\ldots)$. This updated configuration is then used to calculate agent's instant payoffs $\{U^{mp}_i(s_i(t),\Phi_i(t)),\ \ i=1,\ldots,N\}$. This calculation finalizes the time t iteration and the loop continues for the $t+1$ stage.
The formal representation of the simulation procedure is presented in Algorithm 1.

Algorithm 1. Numerical simulations algorithm.

input : N, T, γ, J, σ; `// N supposed to be even`

`// preliminary stage`

calculate Bellman table values $Q_i(s_i(t), \Phi_i(t), t)$

`// game stage`

set initial values for $s_i(0)$ such as $\sum_{i=1}^{N} s_i(0) = 0$

for $t \leftarrow 1$ **to** T **do**

> select random i from $1, \ldots, N$;
>
> generate independent $(\varepsilon_+, \varepsilon_-)$ from $\mathcal{N}(0, \sigma)$;
>
> $m_{-i}(t) = \sum_{k=1..N,\ k\neq i} s_k(t-1)$;
>
> $\Phi_i(t) = (1, s_i(t-1), m_{-i}(t), \varepsilon_+, \varepsilon_-)$;
>
> $U_i^{\mathrm{mp}}(+1, \Phi_i(t)) = \frac{J}{N} m_{-i}(t) + \varepsilon_+$;
>
> $U_i^{\mathrm{mp}}(-1, \Phi_i(t)) = -\frac{J}{N} m_{-i}(t) + \varepsilon_-$;
>
> $V_i(+1|\Phi_i(t), t) = U_i^{\mathrm{mp}}(+1, \Phi_i(t)) + \gamma Q(+1, \Phi_i(t), t)$;
>
> $V_i(-1|\Phi_i(t), t) = U_i^{\mathrm{mp}}(-1|\Phi_i(t)) + \gamma Q(-1, \Phi_i(t), t)$;
>
> **if** $V_i(+1|\Phi_i(t), t) \geq V_i(-1|\Phi_i(t), t)$ **then**
>
> > $s_i(t) = +1$;
>
> **else**
>
> > $s_i(t) = -1$;
>
> **end**
>
> **for** $j \leftarrow 1$ **to** N; $j \neq i$ **do**
>
> > $s_j(t) = s_j(t-1)$;
> >
> > $m_{-j}(t) = \sum_{k=1..N,\ k\neq i,j} s_k(t-1) + s_i(t)$;
> >
> > generate independent $(\varepsilon_+, \varepsilon_-)$ from $\mathcal{N}(0, \sigma)$;
> >
> > $\Phi_j(t) = (1, s_j(t-1), m_{-j}(t), \varepsilon_+, \varepsilon_-)$;
> >
> > $U_j^{\mathrm{mp}}(s_j(t)|\Phi_j(t)) = \frac{J}{N} m_{-j}(t) s_j(t) + \varepsilon_{s_j(t)}$;
>
> **end**

end

output: $s_i(t),\ i = 1, \ldots, N, t = 1, \ldots, T$;

$U_i^{\mathrm{mp}}(s_i(t), \Phi_i(t)),\ i = 1, \ldots, N, t = 1, \ldots, T$

B.2. Examples of system evolution

In this Appendix we provide a detailed illustration of the influence of stochastic cooling on the evolution of the system. As described in the main text, the most dramatic effect we expect to observe is related to the fact that due to this strategic reduction of noise there exists an interval of bare noise strength σ, see Eq. (3), in which the system is pushed into a low-noise phase with $m \neq 0$ at values of σ for which in the absence of strategic effects the system would find itself in the high-noise one with $m = 0$.

To illustrate the effects of the strategic cooling in Fig. B.7 we compare sample evolution patterns of strategy configuration $\mathbf{s}(t) = (s_1(t), \ldots, s_N(t))$ for myopic ($\gamma = 0$, upper row) and strongly farsighted ($\gamma = 0.99$, lower row) for various levels of bare noise σ. The initial configurations $\mathbf{s}(0)$ are in all cases such that $m(0) = 0$. In Fig. B.7 we plot the values $\{s_i(t)\}$ for $i = 1, \ldots N$ (vertical axis) for $t = 0, \ldots, T$ (horizontal axis) with $N = 100$ and $T = 1000$. The empty squares correspond to the values $\{s_i(t) = 1\}$ and the filled ones to the values $\{s_i(t) = -1\}$. it is insightful to compare the sample configuration histories for $\sigma = 1.13$ which for the probit noise and with the assumed value $J = 1$ corresponds to the myopic phase transition point. Comparison of these histories shows that farsighted agents are able to find consensus while myopic ones cannot. At larger bare noise strength $\sigma = 5$ the picture is the same. At very high bare noise level $\sigma = 50$ strategic effects can no longer prevent strategy configuration from randomisation.

The same conclusion can be made from plotting the evolution of $m(t)$ in Fig. B.8.

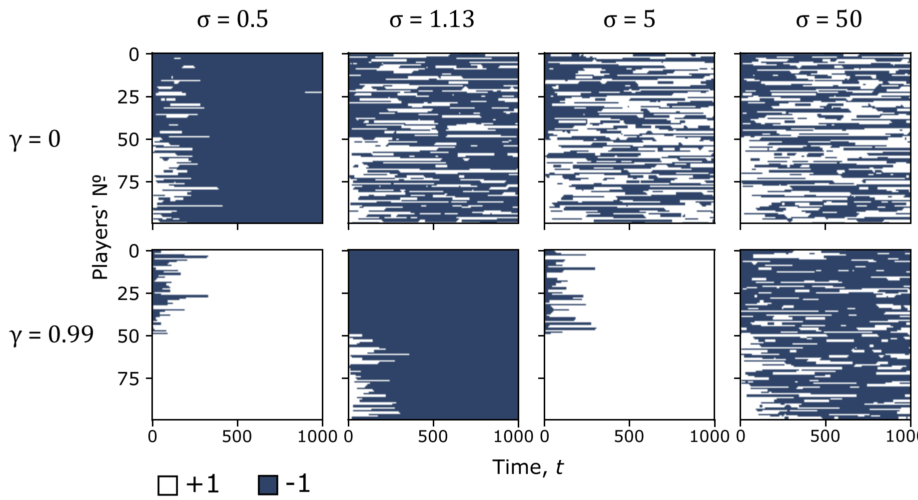

Fig. B.7. Illustration of evolution of configuration $\mathbf{s}(t) = (s_1(t),\ldots,s_N(t))$ for myopic ($\gamma = 0$, upper row) and farsighted ($\gamma = 0.99$, lower row) players for different bare noise levels σ within the time interval $t = 0, \ldots, T$ with $N = 100$ and $T = 1000$. The empty/filled squares correspond to the values $\{s_i(t) = 1\}$ and $\{s_i(t) = -1\}$ respectively.

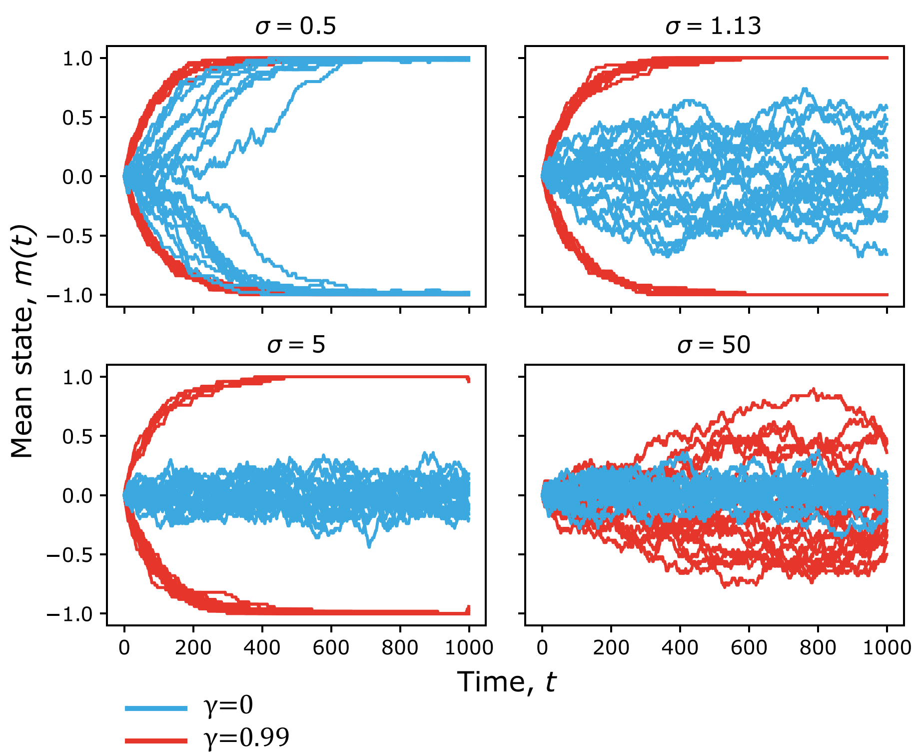

Fig. B.8. Evolution of the agents mean choice in the case of myopic ($\gamma = 0$) and farsighted ($\gamma = 0.99$) games with different bare noise levels. For each pair of γ and σ the results of 20 simulations are shown.

Appendix C. The function $\Delta Q(t,1,m(t))$

In this Appendix we provide details on the behaviour of the function $\Delta Q(t,1,m)$, see (21). In Fig. C.9 we plot $\Delta Q(t,1,m)$ at t equal to 1 and 950 and several values of $\sigma = 1/\beta$ and γ.

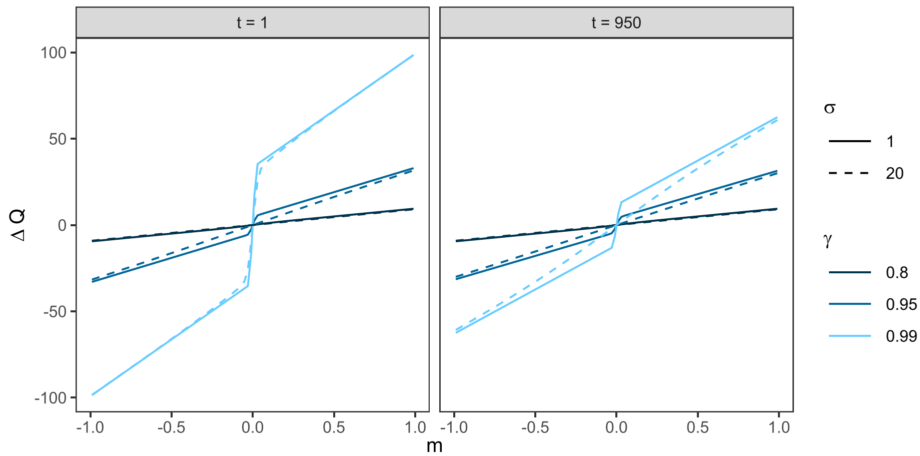

Fig. C.9. The function $\Delta Q(t,1,m)$ for several σ and γ at $t = 1$ and $t = 950$.

From Fig. C.9 and the analysis provided in the Appendix Appendix A we can conclude that the pattern characterising $\Delta Q(t,+1,m)$ can be described as follows:

$$\Delta Q(t,+1,m) \approx d(t,\gamma,\sigma)m + f(t,\gamma,\sigma)\mathrm{erf}(g(t,\gamma,\sigma)m), \tag{C.1}$$

where $d(\cdot)$, $f(\cdot)$ and $g(\cdot)$ are constant for each triple t, γ and σ. However, from Fig. C.9 it is also clear that for each σ starting from some value of γ the shape of the $\Delta Q(t,+1,m)$ function is very close to the linear one:

$$\Delta Q(t,+1,m) \approx c(t,\gamma,\sigma)m, \tag{C.2}$$

with constant $c(\cdot)$ for each triple t, γ and σ. In particular, in Fig. C.9 we see that for $\sigma = 20$ the dependence is linear for $\gamma \le 0.9$, while for $\sigma = 1$ the same is true for $\gamma \le 0.8$.

Let us introduce notations for the following two approximations of the $\Delta Q(t,+1,m)$ function:

$$\Delta Q^{\mathrm{nl}}(t,+1,m) = d(t,\gamma,\sigma)m + f(t,\gamma,\sigma)\mathrm{erf}(g(t,\gamma,\sigma)m),$$
$$\Delta Q^{\mathrm{l}}(t,+1,m) = c(t,\gamma,\sigma)m,$$

where $d(\cdot)$, $f(\cdot)$, $g(\cdot)$ and $c(\cdot)$ are values providing the best fits of the calculated $\Delta Q(t,+1,m)$ function for each (t,γ,σ).

Several ways to quantitatively define the difference between $\Delta Q^{\mathrm{nl}}(t,+1,m)$ and $\Delta Q^{\mathrm{l}}(t,+1,m)$ can be introduced. We will use mean absolute percentage error (MAPE), as it is very natural measure frequently used in statistics to measure forecast accuracy. In particular, let us note $\delta(t,\gamma,\sigma)$ the MAPE between $\Delta Q^{\mathrm{nl}}(t,+1,m)$ and $\Delta Q^{\mathrm{l}}(t,+1,m)$:

$$\delta(t,\gamma,\sigma) = \frac{1}{N} \sum_{k=-\left[\frac{N}{2}\right]}^{\left[\frac{N}{2}\right]} \frac{|\Delta Q^{\mathrm{nl}}(t,+1,\frac{2k}{N}) - \Delta Q^{\mathrm{l}}(t,+1,\frac{2k}{N})|}{|\Delta Q^{\mathrm{nl}}(t,+1,\frac{2k}{N})|} \tag{C.3}$$

In Fig. C.10 the dependence of $\delta(\cdot)$ from γ at $t = 1$ for $\sigma = 1$ and $\sigma = 20$ is presented. From this figure it follows that for γ not too close to $\gamma = 1$ (roughly for $\gamma < 0.85$) the linear fit can be used even for the small σ values.

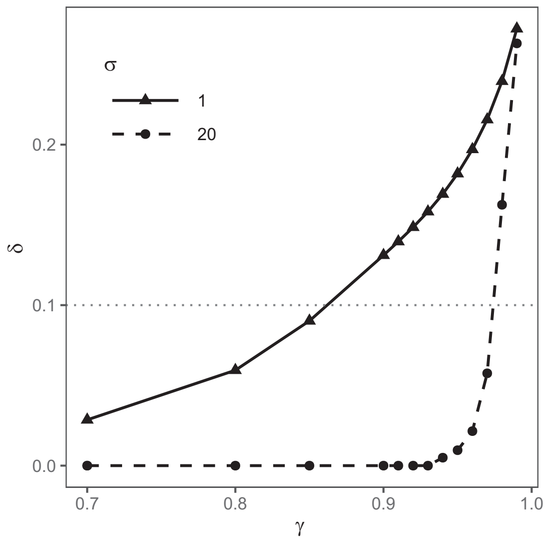

Fig. C.10. The function $\delta(t,\gamma,\sigma)$ for several σ at $t=1$. The dotted horizontal line shows the position of $\delta = 0.1$ – the 10 % MAPE.

The time-dependence of $\Delta Q(t,+1,m)$ in case of linear approximation is determined by that of the function $c(t,\gamma,\sigma)$. The latter is plotted for $\gamma = 0.8$ and several values of σ in the panel A of Fig. C.11. From this figure we see that $c(t,\gamma,\sigma)$ is positive and stays constant for a parametrically large time and rapidly decays to zero at times close to T. Of main interest is, of course, the dominant stationary regime while the decay towards the myopic limit is a boundary effect. In developing a probabilistic description of the effects of far-sightedness we will concentrate on considering the linear regime of (C.2).

The properties of the stationary values of $c(t,\gamma,\sigma)$ are further specified in the panel B of Fig. C.11. Here we see that the coefficient $c(1,\gamma,\sigma)$ is a growing function of the discounting coefficient γ. Another characteristic feature seen in Fig. C.11 is that the coefficient $c(t,\gamma,\sigma)$ is only slightly dependent of the noise scale σ. From the previous note on the dynamics of $c(t,\gamma,\sigma)$ it is clear, that the dependence from γ and σ remains practically unchanged besides the end of the time period.

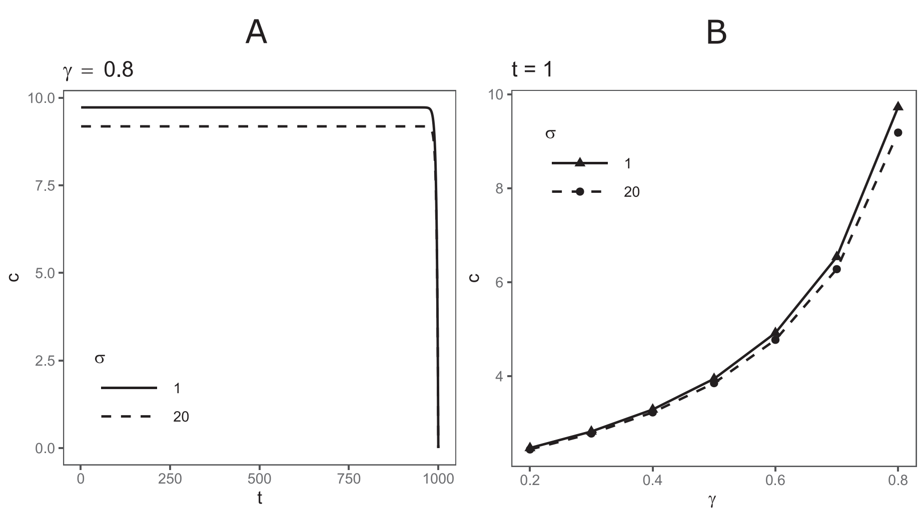

Fig. C.11. Properties of the $c(t,\gamma,\sigma)$ function. Panel **A**: the time-dependence of $c(\cdot)$ for $\gamma = 0.8$ and several σ values. Panel **B**: the dependence of $c(\cdot)$ from γ at $t = 1$ for several σ.

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